Задача №1

а) $\sum\limits_{n=1}^{\infty} (-1)^n n\cdot(e^{\operatorname{tg} 1/n}-1)$ – расх.

$|a_n| = n\cdot(e^{\operatorname{tg} 1/n}-1) \sim 1 \not\to 0 \Rightarrow$ расх. т.к. невыполнено необходимое условие сходимости

б) $\sum\limits_{n=1}^{\infty} \left(\frac{\sqrt{n}+2}{\sqrt{n}+3}\right)^{n^{3/2}}$ – сх.

$\sqrt[n]{a_n} = \left(\frac{\sqrt{n}+2}{\sqrt{n}+3}\right)^{\sqrt{n}} = \left[\left(1+\frac{-1}{\sqrt{n}+3}\right)^{-(\sqrt{n}+3)}\right]^{\frac{-\sqrt{n}}{\sqrt{n}+3}} =$

$= e^{\left(-1+\frac{3}{\sqrt{n}+3}\right)} \xrightarrow[n\to\infty]{} \frac{1}{e} < 1 \Rightarrow$ ряд сх. по пр. Даламбера

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Задача №2

$\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{\cos^2(n^2)}{n^3}$

$\left|(-1)^{n+1}\cdot\frac{\cos^2(n^2)}{n^3}\right| \sim \frac{1}{n^3} \Rightarrow$ сх. ~~но~~ абс. по признаку сравнения в предельной форме

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Задача № 3

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+2n+3}} (x-5)^n$$

при $n \to \infty$ $\quad \sqrt[3]{n^2+2n+3} \sim \sqrt[3]{n^2}$

$$\sqrt[n]{\left|\frac{(-1)^n (x-5)^n}{\sqrt[3]{n^2}}\right|} = \frac{|x-5|}{\underbrace{\left(\sqrt[n]{n}\right)^{\frac{2}{3}}}_{\to 1}} = |x-5|<1$$

$-1 < x-5 < 1$; $\quad 4 < x < 6$

$x=4$: $\quad \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{n^{3/2}} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n^{2/3}}$ — расх, т.к $2/3 < 1$ и ряд знакополож.

$x=6$: $\quad \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1^n}{n^{3/2}}$ — сх. по пр. Лейбница $\left(\frac{1}{n^{3/2}} \to 0\right.$ и ряд монотонно убывает$\left.\right)$

Обл. сх-ти: $x \in (4;6]$

Задача № 5

$$\sum_{n=1}^{\infty} \frac{n^2}{(x+\frac{1}{n})^n} \qquad [2,3]$$

$$\frac{n^2}{(x+\frac{1}{n})^n} \leqslant \frac{n^2}{x^n}; \quad \left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{(n+1)^2 \cdot x^n}{x \cdot x^n \cdot n^2}\right| = \frac{1}{|x|} < 1; \quad |x| > 1$$

Ряд сх. равномерно на промежутке $[-\infty;-1] \cup [1;+\infty) \Rightarrow$

$\Rightarrow$ сх-се и на промежутке $[2;3]$

Задача № 4

а) $f(x) = xe^{x-3}$ $\quad x_0 = 2$

$t = x - x_0 = x - 2; \quad x = t+2$

$$f(x) = (t+2)e^{t-1} = \frac{1}{e}(te^t + 2e^t) = \frac{1}{e}\left(\sum_{n=0}^{\infty}\frac{t^{n+1}}{n!} + \sum_{n=0}^{\infty}\frac{2t^n}{n!}\right) =$$

$$= \{m = n+1\} = \sum_{m=1}^{\infty}\left[\frac{2}{e} + \cancel{t^m}\,\frac{t}{e}^m\left(\frac{1}{(m-1)!} + \frac{2}{m!}\right)\right] =$$

$$= \sum_{m=1}^{\infty}\left[\frac{2}{e} + \frac{2m-1}{e(m-1)m!}(x-2)^m\right]$$

$$\sum_{n=0}^{\infty}\frac{t^{n+1}}{n!} \quad \left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{t\cdot t\cdot \cancel{t^n}\cdot \cancel{n!}}{(n+1)\cancel{n!}\cdot t\cdot \cancel{t^n}}\right| = \frac{|t|}{n+1} \to 0 < 1$$

$$\sum \frac{2t^n}{n!} \quad \left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{2\cdot t\cdot \cancel{t^n}\cdot \cancel{n!}}{(n+1)\cancel{n!}\cdot \cancel{t^n}}\right| = \frac{2|t|}{n+1} \to 0 < 1$$

$\Rightarrow$

$\Longrightarrow$ Обл. сход.: $x \in \mathbb{R}$

$$f^{(102)}(2) = \frac{2\cdot 102 - 1}{101e} = \frac{203}{101}$$

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б) $f(x) = \frac{x}{x^2-3x+2}$ $\quad x_0 = 0$

$$f(x) = \frac{x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \left\{\begin{matrix} x = A(x-1) + B(x-2) \\ A = 2 \quad B = -1 \end{matrix}\right\} =$$

$$= \frac{-1}{1-\frac{x}{2}} + \frac{1}{1-x} = -\sum_{n=0}^{\infty}\frac{x^n}{2^n} + \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}\frac{x^n(2^n-1)}{2^n}$$

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \left|\frac{x\cdot x^n(2\cdot 2^n - 1)\cdot 2^n}{2\cdot 2^n\cdot x^n(2^n-1)}\right| \xrightarrow[n\to\infty]{} x < 1$$

$x = 1$ $\quad \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^n}\right)$ – расх $\Rightarrow$ Обл. сход: $x \in (-\infty; 1)$

$$f^{(102)}(0) = 102!\,(1 - 2^{-102})$$

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№8

$y'' + x^2 y' + 2xy = 1 \quad y(0)=0,\ y'(0)=0 \quad \Delta \le 0{,}001$ при $x \in [0; 0,5]$

$y'' = 1 - x^2 y' - 2xy$; $y''(0) = 1$;

$y''' = -2xy' - x^2 y'' - 2y - 2xy' = -x^2 y'' - 4xy' - 2y$; $y'''(0) = 0$;

$y^{IV} = -2xy'' - x^2 y''' - 4y' - 4xy'' - 2y'' = -2xy'' - x^2 y''' - 2y'' - 4y'$; $y^{IV}(0) = -2$;

$y^{V} = -6y'' - 6xy''' - 2xy''' - x^2 y^{IV} - 2y''' - 4y'' = -x^2 y^{IV} - 8xy''' - 2y''' - 10y''$; $y^{V}(0) = -10$;

$y^{VI} = -2xy^{IV} - x^2 y^{V} - 8y''' - 8xy^{IV} - 2y^{IV} - 10y''' = -x^2 y^{V} - 10xy^{IV} - 2y^{IV} - 18y'''$; $y^{VI}(0) = 4$;

$y(x) = y(0) + y'(0) + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{IV}(0)}{4!}x^4 + \frac{y^{V}(0)}{5!}x^5 + \frac{y^{VI}(0)}{6!}x^6$

$y = \frac{1}{2}x^2 - \frac{2}{4!}x^4 - \frac{10}{5!}x^5 + \frac{4}{6}x^6$

Оценка: $\frac{10}{5!}x^5$ на $[0; 0,5] > 10^{-3}$

$\frac{4}{6!}x^6$ на $[0; 0,5] < 10^{-3}$

Ответ: $y \approx \frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{12}x^5$

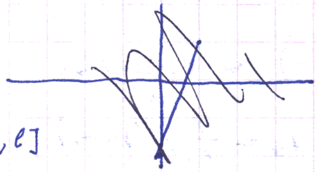
№9

$I = \int_0^{0,1} \frac{\sin x}{x}dx = \int_0^{0,1}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right)dx = \left(x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!}\right)\Big|_0^{0,1} =$

$= \frac{1}{10} - \frac{1}{3\cdot 3!\cdot 10^3} + \frac{1}{5\cdot 5!\cdot 10^5} - \frac{1}{7\cdot 7!\cdot 10^7}$; $|r_2| = \frac{1}{3\cdot 3!\cdot 10^3} < 10^{-4}$

$I \approx 0{,}1$

Задача № 6 $y = 3x - 2$ $(0, 4)$



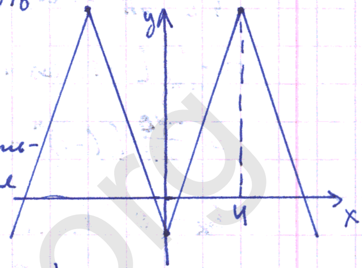
$$f(x) = \frac{a_0}{2} + \sum\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad [0, l]$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l (3x-2)\,dx = \frac{1}{l}\left(\frac{3}{2}x^2 - 2x\right)\Big|_0^l =$$

$$= \frac{3}{2}l - 2 = 6 - 2 = 4$$

а) по ~~синусам~~ косинусам $a_n = \frac{1}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}$

сходимость-поточечная



$$a_n = \frac{1}{4}\int_0^4 (3x-2)\cos\frac{n\pi x}{4} = \frac{1}{4}\int_0^4 \left(3x\cos\frac{n\pi x}{4} - 2\cos\frac{n\pi x}{4}\right) =$$

$$= \frac{1}{4}\left[\cancel{3\sin\cdot x\cdot\sin}\,\frac{4}{n\pi}\cdot 3x\sin\frac{n\pi x}{4} + 3\cos\frac{n\pi x}{4}\cdot\frac{4}{n\pi} - \frac{4\cdot 2}{n\pi}\sin\frac{n\pi}{4}x\right]\Big|_0^4 =$$

$$= \frac{1}{4}\left[\frac{4}{n\pi}\cdot 12\sin n\pi + 3\cos n\pi\cdot\frac{4}{n\pi} - 2\cdot\frac{4}{\pi n}\sin n\pi - 3\cdot\frac{4}{n\pi}\right] =$$

$$= \frac{1}{n\pi}\left[10\sin n\pi + 3\cos n\pi - 3\right] = \frac{1}{n\pi}\left[0 + 3(-1)^n - 3\right] = \frac{3}{n\pi}\left[(-1)^n - 1\right]$$

$$a_{2k} = 0\ ; \quad a_{2k+1} = \frac{-6}{\pi(2k+1)}$$

$$f_1(x) = \cancel{2 + \sum_{n=0}^{\infty}\frac{6}{\pi}\sum_{k=0}\frac{\cos((2k+1)\pi)}{(2k+1)}}\quad 2 + \frac{6}{\pi}\sum_{k=0}^{\infty}\frac{\cos\left(\frac{1}{4}\pi x(2k+1)\right)}{(2k+1)}$$

Равенство Парсеваля $\frac{1}{l}\int_0^l f^2(x)\,dx = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$

$$\frac{1}{l}\int_0^l f^2(x)\,dx = \frac{1}{l}\int_0^l (3x-2)^2\,dx = \frac{1}{12}\int_0^4 (3x-2)^2\,d(3x-2) = \frac{1}{36}(3x-2)^3\Big|_0^4 =$$

$$= \frac{1}{36}\left(\overset{1000}{\cancel{10}} + 8\right) = \cancel{\frac{38}{36}} = \cancel{\frac{19}{18}} = \frac{1008}{36} = 28$$

$$28 = 1 + \sum_{n=1}^{\infty}\frac{36}{\pi^2(2n+1)^2}\ ; \quad 3 = \sum_{n=1}^{\infty}\frac{4}{\pi^2(2n+1)^2}\ ; \quad \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2} = \frac{3\pi^2}{4}$$

б) по синусам $b_n = \frac{1}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{4}\int_0^4 (3x-2)\sin\frac{n\pi x}{4}\,dx = \frac{1}{4}\int_0^4\left(3x\sin\frac{n\pi x}{4} - 2\sin\frac{n\pi x}{4}\right)dx =$

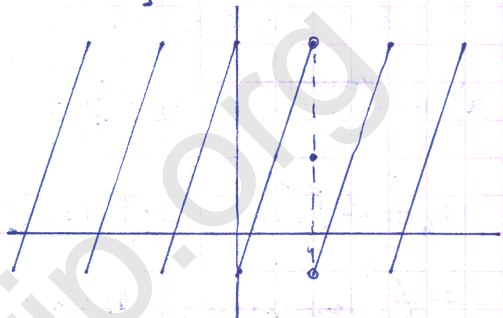
$= \frac{1}{4}\left[-\frac{4}{n\pi}\cdot 3x\cos\frac{n\pi x}{4} + 3\frac{4}{\pi n}\sin\frac{n\pi x}{4} + 2\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right]\Big|_0^4 =$

$= \frac{1}{n\pi}\left[-12\cos n\pi + 3\sin n\pi + 2\cos n\pi + 12 - 2\right] =$

$= \frac{-10}{n\pi}\left((-1)^n - 1\right)$

$b_{2k} = 0 \qquad b_{2k+1} = \frac{20}{\cancel{n}\pi(2k+1)}$

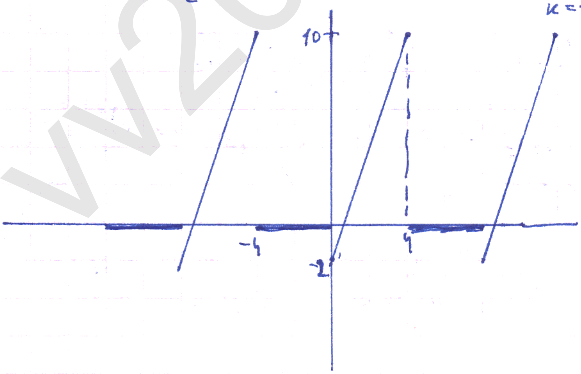
$f_2(x) = \cancel{0} + \frac{20}{\pi}\sum_{k=0}^{\infty}\frac{\sin\left(\frac{1}{4}\pi x(2k+1)\right)}{(2k+1)}$



Сходимость - среднеквадратичная

в) $f_3(x) = \cancel{\frac{\pi}{4} - \frac{x}{4},\ x\in(0,\pi)} \begin{cases} 3x-2, & x\in(0,4) \\ 0, & x\in(-4,0)\end{cases}$

$f_3(x) = \cancel{2}\,\frac{1}{2}\left(f_1(x) + f_2(x)\right) = 1 + \frac{3}{\pi}\sum_{k=0}^{\infty}\frac{\cos\left(\frac{1}{4}\pi x(2k+1)\right)}{(2k+1)} + \frac{10}{\pi}\sum_{k=0}^{\infty}\frac{\sin\left(\frac{1}{4}\pi x(2k+1)\right)}{(2k+1)}$



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Задача № 7. $\frac{\partial^2 u}{d t^2} = \frac{\partial^2 u}{\partial x^2}$, $\ell = 2$

$u(0,t) = u(2,t) = 0$; $u(x,0) = 0$

$\frac{\partial u(x,0)}{dt} = 2x - x^2, \; 0 \le x \le 2$

$$D_n = \frac{2}{2n\pi}\int_0^2 \underbrace{(2x-x^2)}_{u}\,\underbrace{\sin\frac{n\pi}{2}x\,dx}_{dV} = -\frac{2}{n^2\pi^2}(2x-x^2)\cos\frac{n\pi}{2}x\Big|_0^2 +$$

$$+\frac{4}{n^2\pi^2}\int_0^2 (1-x)\cos\frac{n\pi}{2}x\,dx = 0-0+\frac{8}{n^3\pi^3}(1-x)\sin\frac{n\pi}{2}x\Big|_0^2 +$$

$$+\frac{4}{n^2\pi^2}\int_0^2 \sin\frac{n\pi}{2}x\,dx = 0-0-\frac{8}{n^3\pi^3}\cos\frac{n\pi}{2}x\Big|_0^2 = -\frac{8}{n^3\pi^3}\left(\cos(n\pi)-1\right) \ominus$$

$$\cos(n\pi) = \begin{cases} -1, & n = 2k+1 \\ 1, & n = 2k \end{cases} = (-1)^n$$

$$\ominus\; -\frac{8}{n^3\pi^3}\left((-1)^n - 1\right) = +\frac{16}{(2k+1)^3\pi^3}$$

Ответ:

$$u(x,t) = \sum_{k=1}^{\infty} \frac{16}{(2k+1)^3\pi^3}\cdot\sin\left[\frac{(2k+1)\pi}{2}t\right]\cdot\sin\left[\frac{(2k+1)\pi}{2}x\right]$$