Задача № 3

Найти общее решение уравнения

$y'' + ay' + by = f(x)$

используя характеристическое уравнение и метод вариации произвольных постоянных.

$a = -2 \quad b = 1 \quad f(x) = \frac{e^x}{x}$

$y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0 \quad R = 1$ кр 2.

$y_{одн}(x) = C_1 e^x + C_2 x e^x$

$y(x) = C_1(x) e^x + C_2(x) e^x x$

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$$\begin{cases} e^x C_1' + x e^x C_2' = 0 \\ e^x C_1' + C_2'(e^x + x e^x) = \frac{e^x}{x} \end{cases}$$

$\cancel{e^x C_1'} + \cancel{x e^x C_2'} - \cancel{e^x C_1'} - C_2' e^x - \cancel{C_2' e^x x} = -\frac{e^x}{x}$

$C_2' e^x = \frac{e^x}{x} \quad C_2' = \frac{1}{x}$

$C_1' = -C_2' x \Rightarrow C_1' = -1$

$C_1 = -\int dx = -x + D_1$

$C_2 = \int \frac{dx}{x} = \ln|x| + D_2$

$y(x) = (-x + D_1) e^x + (\ln|x| + D_2) x e^x = \tilde{D}_1 e^x + \tilde{D}_2 x e^x + x \ln|x| e^x.$

№5

$L(y) = (x-1)y'' - xy' + y = x^3 - 3x \qquad y_1(x) = x$

Проверим, что $y_1(x) = x$ – частное решение однородного уравнения $(x-1)y'' - xy' + y = 0$.

1) $y_1(x) = x \quad y_1'(x) = 1 \quad y''(x) = 0$

$(x-1)\cdot 0 - x\cdot 1 + x = 0$

$-x + x = 0 \Rightarrow 0 = 0 \Rightarrow$ Верно

Ищем общее решение однородного уравнения в виде

$y_0 = z(x)y_1 = xz \quad y_0' = z + xz' \quad y_0'' = z' + z' + xz'' = 2z' + xz''$

Подставим в уравнение:

$(x-1)(2z' + xz'') - x(z + xz') + xz = 0$

$2(x-1)z' + x(x-1)z'' - xz - x^2z' + xz = 0$

$x(x-1)z'' + z'(2x - 2 - x^2) = 0$

Замена: $z' = u \quad z'' = u'$

$x(x-1)u' + u(2x - 2 - x^2) = 0$

$x(x-1)\frac{du}{dx} = -u(2x - 2 - x^2)$

$\frac{du}{u} = \frac{(x^2 - 2x + 2)dx}{x(x-1)}$

$\ln|u| = \int \frac{(x^2 - x) + x - (2x - 2)}{x^2 - x}dx = x + \ln|x-1| - 2\ln|x| + C =$

$= x + \ln\left|\frac{x-1}{x^2}\right| + C \Rightarrow$

$u = z' = C_1 e^x \frac{x-1}{x^2} = C_1 e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) \Rightarrow$

$z = C_1\int\left(e^x\frac{1}{x} - e^x\frac{1}{x^2}\right)dx = C_1\int\left(e^x\frac{1}{x}\right)'dx = C_1 e^x\frac{1}{x} + C_2 \Rightarrow$

$y_0 = zx = C_1e^x + C_2x$ – общее решение однородного уравнения.

2. Ищем частное решение неоднородного уравнения.

$(x-1)y'' - xy' + y = x^3 - 3x$

$y_{част} = Ax^3 + Bx^2 + \cancel{Cx} + D$ входит в у.од.н.

$y' = 3Ax^2 + 2Bx + C$

$y'' = 6Ax + 2B$

$6Ax^2 + 2Bx - 6Ax - 2B - 3Ax^3 - 2Bx^2 - Cx + Ax^3 + Bx^2 + Cx + D =$

$= 6Ax^2 + 2Bx - 6Ax - 2B - 2Ax^3 - Bx^2 + D = x^3 - 3x$

$-2A = 1 \qquad 2B - 6A = -3$

$6A - B = 0 \qquad -2B + D = 0$

$A = -\frac{1}{2} \quad B = -3 \quad D = -6$

$y_{частн} = -\frac{x^3}{2} - 3x^2 - 6$ – частное решение неоднородного уравнения

$y(x) = C_1e^x + C_2x - \frac{1}{2}x^3 - 3x^2 - 6$ – общее решение неоднород уравнение

№1

$a=2 \quad b=2 \quad T=2a=4$

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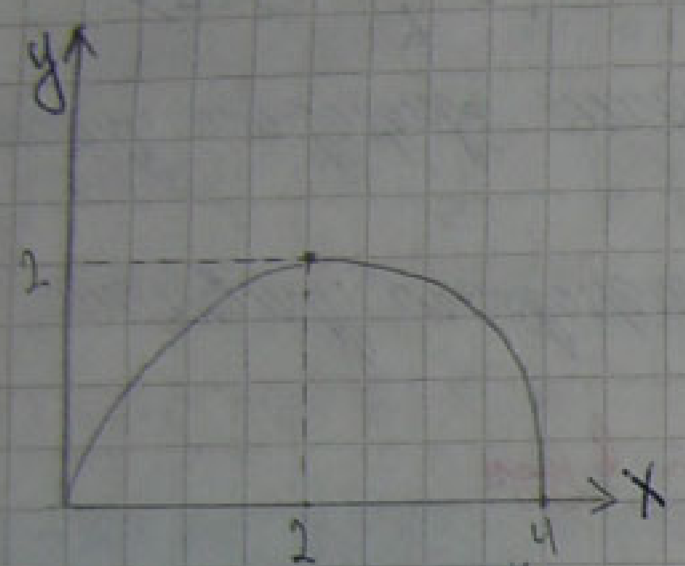
$f(x) = -\frac{1}{2}(x-2)^2+2 = -\frac{1}{2}(x^2-4x)$

$F(p) \doteq \frac{1}{1-e^{-Tp}}\int_0^T e^{-px}f(x)\,dx$ *



$F(p) \doteq \frac{-1}{2(1-e^{-4p})} * \int_0^4 (x^2-4x)e^{-px}dx = \frac{1}{2p(1-e^{-4p})}\int_0^4 (x^2-4x)\,de^{-px} =$

$= \frac{1}{2p(1-e^{-4p})} * \left((x^2-4x)e^{-px}\Big|_0^4 - \int_0^4 (2x-4)e^{-px}dx\right) = \frac{1}{2p(1-e^{-4p})} *$

$\left(0+\frac{1}{p}(2x-4)e^{-px}\Big|_0^4 - 2\int_0^4 e^{-px}dx\right)\Big) = \frac{1}{2p^2(1-e^{-4p})}\left(4e^{-4p}+4+\frac{2}{p}e^{-px}\Big|_0^4\right.$

$= \frac{4}{2p^2(1-e^{-4p})}\left(1+e^{-4p}+\frac{e^{-4p}-1}{2p}\right)$

№2

$y''-2y'+2y = xe^{-x} \quad y(0)=1 \quad y'(0)=1.$

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$Y(x) = y(p)$

$y'(x) = pY(p)-1$

$y''(x) = p^2Y(p)-p+1$

$xe^{-x} = \frac{1}{(p+1)^2}$

$p^2Y(p)-p-1-2pY(p)+2+2Y(p) = \frac{1}{(p+1)^2}$

$Y(p)(p^2-2p+2) = \frac{1}{(p+1)^2}+p-1.$

$Y(p) = \frac{1}{(p+1)^2(p^2-2p+2)} + \frac{p-1}{p^2-2p+2.}$

$\frac{1}{(p+1)^2(p^2-2p+2)} = \frac{A}{p+1}+\frac{B}{(p+1)^2}+\frac{Cp+D}{p^2-2p+2}$

$1 = A(p^2-2p+2)(p+1)+B(p^2-2p+2)+(Cp+D)(p+1)^2$

$p=-1 \quad 1=5B$

$p^3 \quad 0=A+C$

$p^2 \quad 0=A-2A+B+2C+D.$

$p^0 \quad 1=2A+2B+D$

$\begin{cases} B=\frac{1}{5} \quad C=-A. \\ -3A+D=\frac{1}{5} \\ 2A+D=1-\frac{2}{5}=\frac{3}{5} \end{cases}$

$\begin{cases} A=\frac{4}{25} \\ B=\frac{1}{5} \\ C=-\frac{4}{25} \\ D=\frac{7}{25} \end{cases}$

$Y(p) = \frac{4}{25}\cdot\frac{1}{p-1}+\frac{1}{5}\cdot\frac{1}{(p+1)^2}+\frac{1}{25}\cdot\frac{-4p+7}{p^2-2p+2}+\frac{p-1}{p^2-2p+2} = \frac{4}{25}\frac{1}{p+1}+\frac{1}{5}\frac{1}{(p+1)^2}+$

$$\frac{1}{25}\cdot\frac{21p-18}{(p-1)^2+1^2}=\frac{4}{25}\cdot\frac{1}{p+1}+\frac{1}{5}\cdot\frac{1}{(p+1)^2}+\frac{1}{25}\cdot\frac{21(p-1)+3}{(p-1)^2+1^2}=$$

$$=\frac{4}{25}\cdot\frac{1}{p+1}+\frac{1}{5}\cdot\frac{1}{(p+1)^2}+\frac{21}{25}\cdot\frac{p-1}{(p-1)^2+1^2}+\frac{3}{25}\cdot\frac{1}{(p-1)^2+1^2}=$$

$$=\frac{4}{25}e^{-x}+\frac{1}{5}xe^{-x}+\frac{21}{25}e^{x}\cos x+\frac{3}{25}e^{x}\sin x.$$

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в) ✗ №6.

$$\begin{cases}\frac{dx}{dt}=3x-y\\ \frac{dy}{dt}=4x-2y.\end{cases}\qquad x(0)=1\qquad y(0)=-1.$$

$$e^{At}\begin{pmatrix}-1\\-1\end{pmatrix}\doteqdot-(A-pE)^{-1}\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

$$A=\begin{pmatrix}3&-1\\4&-2\end{pmatrix}\qquad A-pE=\begin{pmatrix}3-p&-1\\4&-2-p\end{pmatrix}$$

$$\det(A-pE)=(3-p)(-2-p)+4=p^2-p-2=(p-2)(p+1)$$

$$-(A-pE)^{-1}=\frac{1}{(p-2)(p+1)}\begin{pmatrix}p+2&-1\\4&p-3\end{pmatrix}=\begin{pmatrix}\frac{p+2}{(p-2)(p+1)}&\frac{-1}{(p-2)(p+1)}\\ \frac{4}{(p-2)(p+1)}&\frac{p-3}{(p-2)(p+1)}\end{pmatrix}$$

$$\frac{p+2}{(p-2)(p+1)}=\frac{A}{(p-2)}+\frac{B}{(p+1)}=\frac{4}{3}\frac{1}{p-2}-\frac{1}{3}\cdot\frac{1}{p+1}=\frac{4}{3}e^{2t}-\frac{1}{3}e^{-t}$$

$$(p+2)=A(p+1)+B(p-2)$$

$p=2\qquad A=\frac{4}{3}$

$p=-1\qquad B=-\frac{1}{3}$

$$\frac{4}{(p-2)(p+1)}=\frac{A}{(p-2)}+\frac{B}{(p+1)}=\frac{4}{3}\cdot\frac{1}{p-2}-\frac{4}{3}\frac{1}{p+1}=\frac{4}{3}e^{2t}-\frac{4}{3}e^{-t}$$

$$4=A(p+1)+B(p-2)$$

$p=-1\qquad B=-\frac{4}{3}$

$p=2\qquad A=\frac{4}{3}.$

$$\frac{p-3}{(p-2)(p+1)} = -\frac{1}{3}\frac{1}{p-2} + \frac{4}{3}\frac{1}{p+1} = -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t}$$

$p-3 = A(p+1) + B(p-2)$

$p=-1 \quad B=\frac{4}{3}$

$p=2 \quad A=-\frac{1}{3}$

$$\frac{-1}{(p-2)(p+1)} = \frac{1}{3}\frac{1}{p-2} - \frac{1}{3}\frac{1}{p+1} = \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$$

$-1 = A(p+1) + B(p-2).$

$p=2 \quad A=\frac{1}{3}$

$p=-1 \quad B=-\frac{1}{3}$

$$e^{At} = \begin{pmatrix} \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} \\ \frac{4}{3}e^{2t} - \frac{4}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{At}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3}e^{2t} + \frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ -\frac{4}{3}e^{2t} + \frac{4}{3}e^{-t} + \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{2t} \\ -e^{2t} \end{pmatrix}$$

№4(б)

$y'' - y = \cos x \qquad y(0) = y'(0) = 0$

$\lambda^2 - 1 = 0.$

$\lambda^2 = 1 \qquad +$

$\lambda_{1,2} = \pm 1$

$y_{одн} = C_1 e^{x} + C_2 e^{-x}$

$y_{частн} = A\cos x + B\sin x$

$y' = -A\sin(x) + B\cos(x)$

$y'' = -A\cos(x) + B\sin(x)$

$-A\cos(x) - B\sin(x) - A\cos(x) - B\sin(x) = \cos(x)$

$\cos(x)(-A-A) + \sin(x)(-B-B) = \cos(x)$

$-2A = 1 \qquad A = -\frac{1}{2}$

$-2B = 0 \qquad B = 0.$

$y_{частн} = -\frac{1}{2}\cos(x)$

$y_{он} = -\frac{1}{2}\cos(x) + C_1 e^{x} + C_2 e^{-x}$

1) $y(0) = 0$

$0 = -\frac{1}{2} + C_1 + C_2 \qquad C_1 + C_2 = \frac{1}{2}$

2) $y'(0) = 0$

$y' = \frac{1}{2}\sin x + C_1 e^{x} - C_2 e^{-x}$

$0 = C_1 - C_2 \qquad C_1 = C_2$

$$\begin{cases} c_1+c_2=\frac{1}{2} \\ c_1=c_2 \end{cases} \Rightarrow \begin{cases} 2c_2=\frac{1}{2} \\ c_1=c_2 \end{cases} \Rightarrow \quad c_2=\frac{1}{4} \quad c_1=\frac{1}{4}$$

$$y=-\frac{1}{2}\cos(x)+\frac{e^x}{4}+\frac{e^{-x}}{4}$$

№6б

$$\begin{cases} \frac{dx}{dt}=3x-y & x(0)=-1 \\ \frac{dy}{dt}=4x-2y & y(0)=-1 \end{cases}$$

$x \doteqdot X \qquad \frac{dx}{dt} \doteqdot pX(p)-x_0=pX(p)+1$

$y \doteqdot Y \qquad \frac{dy}{dt} \doteqdot pY(p)-y_0=pY(p)+1$

$$\begin{cases} pX(p)+1=3X-Y \\ pY(p)+1=4X-2Y \end{cases} \Longrightarrow X=\frac{pY(p)+1+2Y}{4}$$

$$\frac{p^2Y(p)}{4}+\frac{p}{4}+\frac{2Yp}{4}+1=\frac{3}{4}pY+\frac{3}{4}+\frac{6Y}{4}-Y$$

$$p^2Y(p)+p-pY+1-2Y=0$$

$$Y(p^2-p-2)=-1-p$$

$$Y(p)=\frac{-(1+p)}{(p+1)(p-2)}=\frac{-1}{p-2}\doteqdot-e^{2t}$$

$$X(p)=\frac{\frac{-p}{p-2}+1-\frac{2}{p-2}}{4}=\frac{\frac{-p+p-2-2}{p-2}}{4}=\frac{\frac{-4}{p-2}}{4}=$$

$$=-\frac{1}{p-2}\doteqdot-e^{2t}$$

$Y(p)=-e^{2t}$

$X(p)=-e^{2t}$

№6а Вариант №4

$$\begin{cases} \frac{dx}{dt} = 3x - y \quad (1) & x_0 = -1 \\ \frac{dy}{dt} = 4x - 2y \quad (2) & y_0 = -1 \end{cases}$$

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$$x = \frac{dy}{4dt} + \frac{1}{2}y$$

$$\frac{dx}{dt} = \frac{1}{4}y'' + \frac{1}{2}y'$$

12.12.07

Подставляем в (1)

6а + 4а +

$$\frac{1}{4}y'' + \frac{1}{2}y' = \frac{3}{4}y' + \frac{3}{2}y - y$$

$$\frac{1}{4}y'' - \frac{1}{4}y' - \frac{1}{2}y = 0.$$

$$y'' - y' - 2y = 0.$$

$$\lambda^2 - \lambda - 2 = 0. \quad D = 1 + 8 = 9 \quad \lambda_{1,2} = \frac{1 \pm \sqrt{9}}{2} \quad \lambda_1 = -1 \quad \lambda_2 = 2$$

$$(\lambda + 1)(\lambda - 2) = 0.$$

$$y_{одн} = c_1 e^{-x} + c_2 e^{2x}.$$

$$y' = -c_1 e^{-x} + 2c_2 e^{2x}$$

$$x = -\frac{1}{4}c_1 e^{-x} + \frac{1}{2}c_2 e^{2x} + \frac{1}{2}c_1 e^{-x} + \frac{1}{2}c_2 e^{2x} = \frac{1}{4}c_1 e^{-x} + c_2 e^{2x}$$

$$\begin{cases} y(0) = -1 = c_1 + c_2 \\ x(0) = -1 = \frac{1}{4}c_1 + c_2 \end{cases} \quad c_2 = -1 - c_1 \quad -1 - \frac{1}{4}c_1 + 1 + c_1 = 0 \quad \frac{3}{4}c_1 = 0. \quad c_1 = 0. \quad c_2 = -1.$$

$$\begin{cases} x = -e^{2t} \\ y = -e^{2t} \end{cases}$$

№4а $\quad y'' - y = \cos x \quad y(0)=0 \quad y'(0)=0$

Решим вспомогательную задачу Коши

$z'' - z = 1 \quad z(0)=0 \quad z'(0)=0$

$z \risingdotseq Z$

$z'' \risingdotseq p^2 Z$

$p^2 Z - Z = 1$

$$Z = \frac{1}{p^2-1} = \frac{A}{p-1} + \frac{B}{p+1}$$

$Ap + A + Bp - B = 1$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \quad A=\frac{1}{2} \quad B=-\frac{1}{2}$$

$$Z = \frac{1}{2}\left(\frac{1}{p-1}\right) - \frac{1}{2}\left(\frac{1}{p+1}\right) \risingdotseq z = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

С помощью формулы Дюамеля

$$y = f * z' = \cos x * \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right) = \int_0^x \cos t \cdot \left(\frac{1}{2}e^{x-t} - \frac{1}{2}e^{-x+t}\right)dt =$$

$$= \frac{1}{2}e^x \cdot \int_0^x \cos t \cdot e^{-t}dt - \frac{1}{2}e^{-x}\int_0^x \cos t\, e^t dt =$$

$$I_1 = \int_0^x \cos t\, e^{-t}dt = \begin{bmatrix} u=\cos t & du=\sin t\,dt \\ dv=e^{-t}dt & v=-e^{-t} \end{bmatrix} = -\cos t\, e^{-t}\Big|_0^x - \int_0^x \sin t\, e^{-t}dt =$$

$$= -\cos x\, e^{-x} + 1 - \int_0^x \sin t\, e^{-t}dt = \begin{bmatrix} u=\sin t & du=\cos t\,dt \\ dv=e^{-t}dt & v=-e^{-t} \end{bmatrix} =$$

$$= -\cos x\, e^{-x} + 1 + e^{-t}\sin t\Big|_0^x - \int_0^x \cos t\, e^{-t}dt = -\cos x \cdot e^{-x} + 1 + e^{-x}\sin x - I_1$$

$$I_1 = \frac{1}{2}\left(-\cos x\, e^{-x} + 1 + e^{-x}\sin x\right)$$

$$I_2 = \int_0^x \cos t\, e^t dt = \cos t\, e^t\Big|_0^x + \int_0^x \sin t\, e^t dt = e^x\cos x - 1 + e^t\sin t\Big|_0^x -$$

$$- \int_0^x \cos t\, e^t dt = e^x\cos x - 1 + e^x\sin x - I_2$$

$$I_2 = \frac{1}{2}\left(e^x\cos x - 1 + e^x\sin x\right)$$

$$y = \frac{1}{2}e^x \cdot I_1 - \frac{1}{2}e^{-x}I_2 = -\frac{1}{4}\cos x + \frac{1}{4}e^x + \frac{1}{4}\sin x -$$

$$- \frac{1}{4}\cos x + \frac{1}{4}e^{-x} - \frac{1}{4}\sin x = -\frac{1}{2}\cos x + \frac{1}{4}e^x + \frac{1}{4}e^{-x}$$